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NOTE

Relativistic Combination of Any Number of Collinear Velocities and Generalization of Einstein's Formula

Raymond and Beverly Sackler Faculty of Exact Sciences, Department of Geophysics and Planetary Sciences, Tel Aviv University, Tel Aviv 69978, Israel

and

Enders A. Robinson

Maurice Ewing and J. Lamar Worzel Professor of Geophysics, Columbia University, New York, New York 10027

Submitted by George Leitmann

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By a direct application of the theory of polynomials orthogonal on the unit circle, the Einstein relativistic combination of any number of velocities is given in a surprisingly simple closed form. By doing so the Antippa additive formula is simplified. Insight is gained through treatment of the wave equations describing pressure and particle velocities for a layered model. © 2000 Academic Press

In a recent article by Antippa [1], a closed form expression of the relativistic velocity addition formula known as the Einstein law [2] was developed. Here we demonstrate that even simpler relations hold.

Let $\beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_n$ denote the relativistic addition of n collinear scaled velocities $\beta_1, \beta_2, \ldots, \beta_n$. The scaling is such that the velocity of light in empty space is unity. We assume that the velocities are measured in units for which the velocity of light is unity, since then each of the velocities is less than one in magnitude, in accordance with Einstein's

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famous principle. Antippa [1] gives the additive formula for the Einstein law

$$\beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_n = \frac{\sum_{m=0,\, m=\text{odd}}^{n} \sum_{l_1+l_2+l_n=m,\, l_j \varepsilon (0,1)} \prod_{j=1}^{n} \beta_j^{l_j}}{\sum_{m=0,\, m\ \text{even}}^{n} \sum_{l_1+l_2+l_n=m,\, l_j \varepsilon (0,1)} \prod_{j=1}^{n} \beta_j^{l_j}} \quad (1)$$

which holds for any number of collinear velocities. In this paper we simplify Antippa's formula. We make use of the classical theory of polynomials orthogonal on the unit circle. The general theory of polynomials orthogonal on the unit circle was developed by Szego [11] and Smirnov [10]. The general case was treated in the book by Geronimus [3]. These polynomials are the backbone of the analytic solution of the wave equation (see, for instance, Loewenthal and Stoffa [5]).

The polynomials for $1, 2, 3, \ldots, n$ are defined in terms of a set of real numbers $\beta_1, \beta_2, \ldots, \beta_n$ each of which is less than one in magnitude. The polynomials of the first kind satisfy the Schur–Levinson recursion [4, 9].

$$\begin{aligned} A_k(Z, \beta_1, \beta_2, \ldots, \beta_k) &= A_{k-1}(Z, \beta_1, \beta_2, \ldots, \beta_{k-1}) \\ &\quad + \beta_k Z^k A_{k-1}\big(Z^{-1}, \beta_1, \beta_2, \ldots, \beta_{k-1}\big), \end{aligned} \quad (2)$$

whereas the polynomials of the second kind satisfy the Schur–Levinson recursion

$$\begin{aligned} B_k(Z, \beta_1, \beta_2, \ldots, \beta_k) &= B_{k-1}(Z, \beta_1, \beta_2, \ldots, \beta_{k-1}) \\ &\quad - \beta_k Z^k B_{k-1}\big(Z^{-1}, \beta_1, \beta_2, \ldots, \beta_{k-1}\big), \end{aligned} \quad (3)$$

where

$$k = 1, 2, \ldots, n \qquad \text{and} \qquad A_0(Z) = B_0(Z) = 1. \quad (4)$$

Note that A_k and B_k, are polynomials of the Kth degree in Z and represent time series in the Z transform domain.

As an aside, we give the following motivation for our results. The orthogonal polynomials find use in the theory of the direct and multiple reflections of plane acoustic waves occurring in layered media. The real numbers $\beta_1, \beta_2, \ldots, \beta_n$ represent the Fresnel reflection coefficients of the successive interfaces in isolation from the other interfaces. The polynomials A_k and B_k represent, respectively, the pressure field and particle velocity seismograms as measured by hydrophones and geophones at the Kth layers. For more details the reader is referred to Loewenthal and Robinson [6]. The system reflection coefficient R_n can be expressed in terms of the two fundamental polynomials as

$$R_n(Z, \beta_1, \dots, \beta_2, \dots, \beta_n) \\ = \frac{A_n(Z, \beta_1, \beta_2, \dots, \beta_n) - B_n(Z, \beta_1, \beta_2, \dots, \beta_n)}{A_n(Z, \beta_1, \beta_2, \dots, \beta_n) + B_n(Z, \beta_1, \beta_2, \dots, \beta_n)}. \tag{5}$$

Robinson [7, 8] has shown that when $Z = 1$, the system reflection coefficient for two interfaces gives the classical Einstein addition of the reflection coefficients; that is,

$$R_2(1, \beta_1, \beta_2) = \beta_1 \oplus \beta_2 = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}. \tag{6}$$

A similar observation was made in optics by Vigoureux [12].

The extension of this result is

$$R_n(1, \beta_1, \beta_2, \dots, \beta_n) = \beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_n. \tag{7}$$

The generalized impedance of the system can be expressed as

$$I_n(Z, \beta_1, \beta_2, \dots, \beta_n) = \frac{1 - R_n(Z, \beta_1, \beta_2, \dots, \beta_n)}{1 + R_n(Z, \beta_1, \beta_2, \dots, \beta_n)}. \tag{8}$$

Substituting Eq. (5) into Eq. (8), we obtain

$$I_n(Z, \beta_1, \beta_2, \dots, \beta_n) = \frac{B_n(Z, \beta_1, \beta_2, \dots, \beta_n)}{A_n(Z, \beta_1, \beta_2, \dots, \beta_n)}. \tag{9}$$

Setting $Z = 1$ in Eq. (5) we obtain the result, given by Loewenthal and Stoffa [5], for surface reflection coefficient $\beta_0 = 0$,

$$R_n(1, \beta_1, \beta_2, \dots, \beta_n) \\ = \frac{(1 + \beta_n)(1 + \beta_{n-1}) \cdots (1 + \beta_1) - (1 - \beta_n)(1 - \beta_{n-1}) \cdots (1 - \beta_1)}{(1 + \beta_n)(1 + \beta_{n-1}) \cdots (1 + \beta_1) + (1 - \beta_n)(1 - \beta_{n-1}) \cdots (1 - \beta_1)}. \tag{10}$$

Because

$$\begin{aligned} A_n(1, \beta_1, \beta_2, \dots, \beta_n) & \\ &= A_{n-1}(1, \beta_1, \beta_2, \dots, \beta_{n-1}) + \beta_k A_{n-1}(1, \beta_1, \beta_2, \dots, \beta_{n-1}) \\ &= (1 + \beta_n) A_{n-1}(1, \beta_1, \beta_2, \dots, \beta_{n-1}) \\ &= (1 + \beta_n)(1 + \beta_{n-1}) \cdots (1 + \beta_1), \end{aligned} \tag{11}$$

and similarly,

$$B_n(1, \beta_1, \beta_2, \ldots, \beta_n) = (1 - \beta_n)(1 - \beta_{n-1}) \cdots (1 - \beta_1), \tag{12}$$

Eqs. (7) and (10) give

$$\beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_n = \frac{\prod_{j=1}^{n}(1 + \beta_j) - \prod_{j=1}^{n}(1 - \beta_j)}{\prod_{j=1}^{n}(1 + \beta_j) + \prod_{j=1}^{n}(1 - \beta_j)}, \tag{13}$$

which represents an essential simplification of Antippa's expression (1). For example, when $n = 2$ Antippa's expression gives

$$\begin{aligned} \beta_1 \oplus \beta_2 &= \frac{\sum_{l_1+l_2=1,\, l_j \varepsilon (0.,)} \prod_{j=1}^{2} \beta_j^{l_j}}{\sum_{l_1=0,\, l_2=0} \prod_{j=1}^{n} \beta_j^{l_j} + \sum_{l_1+l_2=2,\, l_j \varepsilon (0,1)} \prod_{j=1}^{n} \beta_j^{l_j}} \\ &= \frac{\beta_1^1 \beta_2^0 + \beta_1^0 \beta_2^1}{\beta_1^0 \beta_2^0 + \beta_1^1 \beta_2^1} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}, \end{aligned} \tag{14}$$

whereas our expression (14) gives

$$\beta_1 \oplus \beta_2 = \frac{(1 + \beta_1(1 + \beta_2) - (1 - \beta_1)(1 - \beta_2)}{(1 + \beta_1)(1 + \beta_2) + (1 - \beta_1)(1 - \beta_2)} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{15}$$

which is the same.

Note that Eqs. (5) and (7) generalize Eq. (13), which is just the low frequency approximation to the time series for A and B. This will be discussed in a forthcoming work.

In conclusion, the relativistic combination of any number of collinear velocities as a closed form expression can be obtained by a direct application of the theory of polynomials orthogonal on the unit circle. A surprisingly simple formula is obtained. Moreover, the use of this approach with an unspecified value of the parameter Z can provide valuable insight into the basic underlying structure of Einstein's theory of relativity.

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